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NOTE

Defining Wave Amplitude in Characteristic Boundary Conditions

Key Words: Euler compressible equations; characteristic boundary conditions; nonreflecting conditions; initial conditions.

Characteristic treatment of boundary conditions for the Euler equations relies on determining the strength of the waves entering the computational domain as a function of the strength of the outgoing waves and the physical boundary conditions. The purpose of this note is to demonstrate how critical the definition chosen for the wave amplitudes can be.

The 2D Euler equations may be expressed in quasi-linear form as

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{V}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{V}}{\partial y} = \mathbf{0}. \tag{1}$$

Here $\mathbf{V} = (\rho, u, v, P)^{\mathrm{T}}$ is the vector of primitive variables and each of the matrices $\mathbf{A}$ and $\mathbf{B}$ has its own complete set of real eigenvalues and right and left eigenvectors. The matrix $\mathbf{E_n}$ defined as $\mathbf{A}n_x + \mathbf{B}n_y$ can be introduced, where $\mathbf{n}$ is chosen as the outward normal to the boundary under consideration. By diagonalizing $\mathbf{E_n}$ the eigenvalue matrix

$$\mathbf{\Lambda_n} = \mathbf{L_n}\mathbf{E_n}\mathbf{L_n^{-1}} = \mathrm{diag}\left(\lambda_n^1, \lambda_n^2, \lambda_n^3, \lambda_n^4\right) = \mathrm{diag}(u_n, u_n, u_n + c, u_n - c), \tag{2}$$

is obtained, where $u_n = \mathbf{u}\cdot\mathbf{n}$ and c is the speed of sound. The matrices $\mathbf{L_n}$ ($\mathbf{L_n^{-1}}$) with left (right) eigenvectors as rows (columns) relate variations in the characteristic variables $\mathbf{W_n}$ to variations in the primitive vector $\mathbf{V}$ through the relations

$$\delta\mathbf{W_n} = \mathbf{L_n}\delta\mathbf{V}, \quad \delta\mathbf{V} = \mathbf{L_n^{-1}}\delta\mathbf{W_n}. \tag{3}$$

In 2D, the four characteristic variables satisfy a set of convection equations with the speed of propagation given by Eq. (2), with source terms related to pressure and velocity variations in the $\mathbf{s}$-direction, where $\mathbf{s}$ forms an orthonormal basis $(\mathbf{n}, \mathbf{s})$ with $\mathbf{n}$. These equations are

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obtained by premultiplying Eq. (1) by $\mathbf{L_n}$. The fourth equation reads

$$\frac{\partial W_n^4}{\partial t} + (\mathbf{u} - c\mathbf{n}) \cdot \nabla W_n^4 + c\mathbf{s} \cdot \nabla W_n^2 = 0. \tag{4}$$

Applying an explicit Euler time discretization to Eq. (1), the update of primitive variables can be written as

$$\Delta \mathbf{V} = \mathbf{V}^{n+1} - \mathbf{V}^n = -\Delta t \mathcal{R} = -\Delta t \left[\mathbf{A} \frac{\partial \mathbf{V}}{\partial x} + \mathbf{B} \frac{\partial \mathbf{V}}{\partial y} \right]. \tag{5}$$

For a given boundary with normal $\mathbf{n}$, the full residual $\mathcal{R}$ of Eq. (5) can be split into a normal component $\mathcal{R}_n$ (involving only normal derivatives) and a tangential component $\mathcal{R}_s$ (involving only derivatives along $\mathbf{s}$). Let us define $\mathbf{V}^n$ as the boundary value at time level n, and $\delta \mathbf{V}^P$ the predicted boundary update from the interior scheme, prior to application of the boundary condition. Define also $\delta \mathbf{V}_w^P$ as the component of $\delta \mathbf{V}^P$ to which the boundary condition will be applied, and $\delta \mathbf{V}^U = \delta \mathbf{V}^P - \delta \mathbf{V}_w^P$ as the part of the boundary update which is not affected by the characteristic boundary condition. Typically, a characteristic based boundary treatment is applied as follows:

1. Choose the part of the residual ($\delta \mathbf{V}_w^P$) to which the boundary conditions are to be applied. If it is the complete residual, $\delta \mathbf{V}^U = 0$.
2. From Eq. (3), decompose $\delta \mathbf{V}_w^P$, into characteristic variations $\delta \boldsymbol{W}_n^{\text{in},P}$ and $\delta \boldsymbol{W}_n^{\text{out}}$ due to ingoing and outgoing waves, with corresponding primitive variations $\delta \mathbf{V}_w^{\text{in},P}$ and $\delta \mathbf{V}_w^{\text{out}}$.
3. Modify the amplitude of the incoming wave(s) $\delta \boldsymbol{W}_n^{\text{in},P}$ according to the physical requirements at the boundary. This produces the corrected amplitudes, $\delta \boldsymbol{W}_n^{\text{in},C}$. Retain the outgoing waves $\delta \boldsymbol{W}_n^{\text{out}}$ or $\delta \mathbf{V}_w^{\text{out}}$ as they are.
4. Combine the waves $\delta \boldsymbol{W}_n^{\text{in},C}$ and $\delta \boldsymbol{W}_n^{\text{out}}$, and using $\mathbf{L_n^{-1}}$, Eq. (3), transform back to primitive variables. This gives $\delta \mathbf{V}_w^C$. The boundary point is then updated as

$$\mathbf{V}^{n+1} = \mathbf{V}^n + \delta \mathbf{V}^U + \delta \mathbf{V}_w^C = \mathbf{V}^n + \delta \mathbf{V}^U + \delta \mathbf{V}^{\text{in},C} + \delta \mathbf{V}^{\text{out}}.$$

The decomposition of the Euler equations into a set of waves traveling normally to the boundary provides a theoretical basis to derive proper boundary condition treatments, following steps 2–3–4 above. However, such theory gives no indication of the best definition for the part of the update to which the boundary conditions are to be applied. This is the principal reason why so many different formulations are discussed in the literature [1–8].

Let us define an approach, we shall call the *full residual* approach, as a boundary treatment such that in step 1 above $\delta \mathbf{V}_w^P = -\Delta t \mathcal{R}$. Following steps 2–4, in the case of a 2D subsonic outlet, this leads to the nonreflecting boundary condition

$$\frac{\partial W^4}{\partial t} = 0 = -\frac{\partial u_n}{\partial t} + \frac{1}{\rho c} \frac{\partial P}{\partial t} \tag{6}$$

which is equivalent to that proposed in [1–4].

Similarly, we shall define the *normal* approach as the boundary treatment such that for step 1, $\delta \mathbf{V}_w^P = -\Delta t \mathcal{R}_n$. This leads to the nonreflecting condition

$$(u_n - c) \frac{\partial W^4}{\partial n} = 0 = (u_n - c) \left[-\frac{\partial u_n}{\partial n} + \frac{1}{\rho c} \frac{\partial P}{\partial n} \right] \tag{7}$$

which is equivalent to the forms in [5, 6], although presented in a completely different formalism. Hirsch [7] argues that the nonreflecting condition has to be applied to the advection terms of the bicharacteristic equations

$$(u_n - c)\frac{\partial W^4}{\partial n} + u_s \frac{\partial W^4}{\partial s} = 0. \tag{8}$$

Following Giles [8], the analysis of the linearized Euler equations based on a Fourier decomposition of the solution at the boundary gives

$$\frac{\partial W^4}{\partial t} = -u_n \frac{\partial W^2}{\partial s} - u_s \frac{\partial W^4}{\partial s}; \tag{9}$$

Eqs. (6), (7), (8), and (9) are all nonreflecting boundary conditions based on characteristic analysis. In 1D, they all reduce to one of the forms $(u_n - c)\ (\partial W^4/\partial n) = 0$ or $\partial W^4/\partial t = 0$ which are equivalent since the last characteristic equation is simply $\partial W^4/\partial t + (u_n - c)$ $(\partial W^4/\partial n)$ in this case. However, these boundary treatments are not equivalent in 2D. It transpires that under certain circumstances they can even produce completely different results.

For example, consider the computation domain defined spatially by $0 < x < 1$ and $0 < y < 1$. The initial condition is uniform for the density and the static pressure, and zero for the velocity in the y-direction. For the streamwise velocity, we impose $u(x, y) = U_0(1.5 + \tanh(10(y - 0.5)))$ for $x = 0$ and $u(y) = 0$ elsewhere. U_0 is chosen such that the flow is subsonic everywhere. The *full residual* approach is used at the inlet to impose the velocity components and the temperature while a nonreflecting condition is tested at the outlet. The *normal* nonreflecting characteristic condition is used for both $y = 0$ and $y = 1$ to allow acoustic disturbances in the y-direction to leave the domain. The velocity profile is expected to propagate downstream during the computation. The steady solution is obviously $u(x, y) = u(0, y)$ for all x. Typical velocity profiles obtained after convergence with both the *full residual* and *normal* formulations at the outlet boundary are shown in Fig. 1.

Clearly the *full residual* outlet condition prevents the given velocity profile from propagating along the x-direction. Instead, the u-velocity tends to be uniform near the exit. On the other hand, the use of the *normal* approach leads to the correct velocity profiles. Both the Hirsh and the Giles formulations allow the hyperbolic tangent profile to propagate as

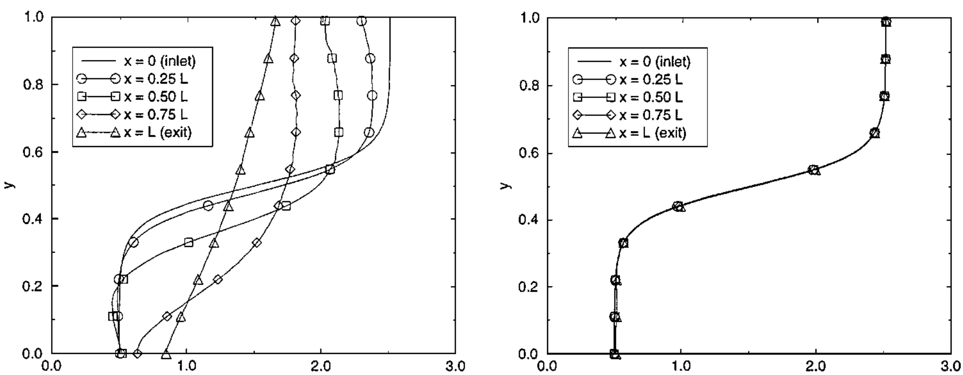

FIG. 1. Velocity profiles at different abscissa for the full residual (left) and the normal (right) non-reflecting outlet boundary condition.

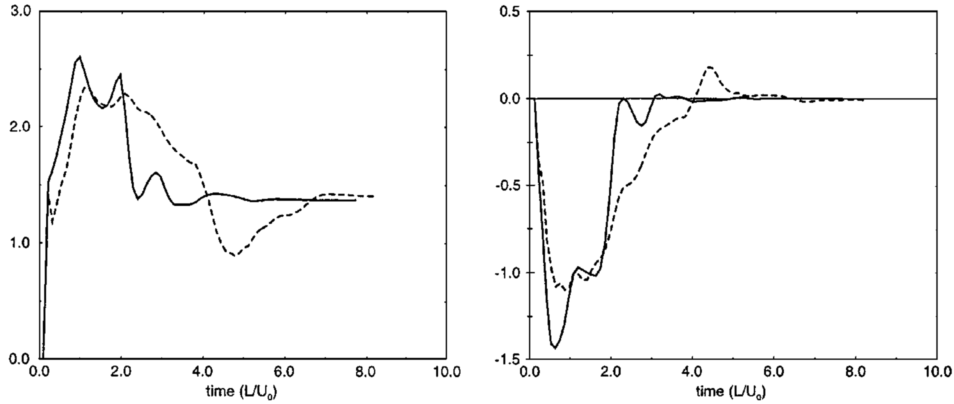

FIG. 2. Time evolutions of the streamwise (left) and the normal (right) velocity component at a point at the outlet boundary for the Giles (dashed line) and *normal* (solid line) formulations.

expected, but the relaxation time to the steady state is longer with the Giles condition (see Fig. 2).

These dramatic differences may be explained by formulating all the boundary conditions in the same framework. Any (nonreflecting) boundary condition can be written either in terms of time derivatives (*temporal form*) or in terms of normal derivatives (*spatial form*). These two forms are linked through the compatibility relation, Eq. (4), and imposing a boundary condition on the time derivative can be translated into a condition on the normal derivative and vice versa. An overview of the various conditions considered, written in their two equivalent forms (*temporal* and *spatial*), is given in Table I.

This table provides a formal comparison of these boundary conditions. Of course, the results of a computation depends only on the choice of the boundary (the rows in the table) and *not* on the form under which it is written (the columns in the table). The *full residual* formulation imposes $\partial W_n^4/\partial t = 0$ and thus forces the temporal evolutions of the streamwise velocity and the pressure to remain nearly proportional. At the initial time, both quantities are uniform at the exit, so that their profiles keep the same shape during the computation if $1/\rho c$ does not depend on y. This feature of the solution at the boundary is well predicted by the computation (not shown) but is not compatible with the present physical configuration. Actually the full residual approach is likely to give incorrect results as soon as the initial conditions at the boundary are inconsistent with the actual flow structure. The *temporal* forms of the other three boundary treatments (see Table I) show that the temporal evolution

TABLE I
Correspondence between the *Temporal* and the *Spatial* Form for Some Nonreflecting Boundary Conditions; 2D Case

Name	Temporal form	Spatial form
Thompson [4]	$\frac{\partial W^4}{\partial t} = 0$, Eq. (6)	$\frac{\partial W^4}{\partial n} = -\frac{1}{u_n - c}\left\{u_s \frac{\partial W^4}{\partial s} + c\frac{\partial W^2}{\partial s}\right\}$
Poinsot [6]	$\frac{\partial W^4}{\partial t} = -\left\{u_s \frac{\partial W^4}{\partial s} + c\frac{\partial W^2}{\partial s}\right\}$	$\frac{\partial W^4}{\partial n} = 0$, Eq. (7)
Hirsh [7]	$\frac{\partial W^4}{\partial t} = -c\frac{\partial W^2}{\partial s}$	$\frac{\partial W^4}{\partial n} = -\frac{u_s}{(u_n - c)}\frac{\partial W^4}{\partial s}$, Eq. (8)
Giles [8]	$\frac{\partial W^4}{\partial t} = -u_n\frac{\partial W^2}{\partial s} - u_s\frac{\partial W^4}{\partial s}$, Eq. (9)	$\frac{\partial W^4}{\partial n} = \frac{\partial W^2}{\partial s}$

of the velocity and the pressure are no longer proportional. Three different terms appear in the right-hand side of the *temporal forms*, namely $A = u_s(\partial W^4/\partial s)$, $B = c(\partial W^2/\partial s)$ and $C = u_n(\partial W^2/\partial s)$. At least one term is needed to ensure that the given boundary condition can reach the correct steady state. For Hirsh's condition which leads to results that are equivalent to the *normal* approach, the term denoted above as A is not critical (this term is the difference between two boundary conditions which give the same results; see Table I). Thus the term B is responsible for the success of the computations with those conditions. One observes also that the term C in the Giles treatment is nothing but a factor of u_n/c smaller than B in the present subsonic test case. Accordingly, with the Giles treatment the relaxation time to the steady state has been found to be loner than for the other conditions (see Fig. 2). More details of the present study are available in [9], including the 3D version of Table I and implementation details of the boundary treatments for a flow solver based on hybrid meshes. Some preliminary runs were performed by Dr. G. Hernandez. All the computations were done with the Fortran library AVBP/COUPL developed at CERFACS.

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